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Approximate Wavelets and the Approximation of Pseudodifferential Operators

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This paper studies *approximate multiresolution analysis* for spaces generated by smooth functions providing high-order semi-analytic cubature formulas for multi-dimensional integral operators of mathematical physics. Since these functions satisfy refinement equations with any prescribed accuracy, methods from wavelet theory can be applied. We obtain an approximate decomposition of the finest scale space into almost orthogonal wavelet spaces. For the example of the Gaussian function we study some properties of the analytic prewavelets and describe the projection operators onto the wavelet spaces. The multivariate wavelets retain the property of the scaling function to provide efficient analytic expressions for the action of important integral operators, which leads to sparse and semi-analytic representations of these operators.

1. INTRODUCTION

In this paper we introduce the so-called *approximate* wavelet decompositions of spaces of approximating functions which appear to be useful for constructing high-order semi-analytic cubature formulas for important classes of pseudodifferential and other integral operators of mathematical physics. These functions satisfy refinement equations not exactly, but in some approximate sense, which allows one to decompose within some given tolerance fine scale spaces into a direct sum of wavelet spaces and thus to derive sparse representations of these operators.

The application of wavelet-based methods to the representation of integral and differential operators is one of the actual research topics in the numerical analysis of solution

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methods for the corresponding operator equations. The usual setting is based on multiresolution analysis, introduced in [11, 17]. Starting from a finite sequence of nested closed subspaces

$$V_0 \subset V_1 \subset \cdots \subset V_n \subset L_2(\mathbf{R}^d), \tag{1.1}$$

the space of approximating functions V_n, corresponding to the finest grid, is decomposed into the orthogonal sum

$$V_n = V_0 \bigoplus_{j=0}^{n-1} W_j, \tag{1.2}$$

where the wavelet space W_j is the orthogonal complement $W_j = V_{j+1} \ominus V_j$. The chain (1.1) is called a (stationary) multiresolution analysis of V_n if the spaces V_j have the properties

(i) $f(\mathbf{x}) \in V_0$ if and only if $f(\mathbf{x} - \mathbf{m}) \in V_0$ for any $\mathbf{m} \in \mathbf{Z}^d$;
(ii) $f(\mathbf{x}) \in V_j$ if and only if $f(2\mathbf{x}) \in V_{j+1}$ for any $j = 0, \ldots, n-1$;
(iii) there exists a function ϕ such that $\{\phi(\cdot - \mathbf{m})\}_{\mathbf{m}\in\mathbf{Z}^d}$ is an L_2-stable basis in V_0, i.e., there exist constants $c_2 > c_1 > 0$ such that

$$c_1 \|\{a_\mathbf{m}\}\|_{l^2} \le \Big\| \sum_{\mathbf{m}\in\mathbf{Z}^d} a_\mathbf{m}\phi(\cdot - \mathbf{m})\Big\|_2 \le c_2 \|\{a_\mathbf{m}\}\|_{l_2}, \qquad \forall \{a_\mathbf{m}\} \in l_2(\mathbf{Z}^d). \tag{1.3}$$

Then obviously the spaces V_j are spanned by the dilated shifts $\phi(2^j \cdot - \mathbf{m})$, $\mathbf{m} \in \mathbf{Z}^d$, of the scaling function ϕ. The main goal of the multiresolution is to determine a new basis of the space V_n, which is used in numerical procedures. It is well known that there exist $2^d - 1$ functions $\psi_\nu \in W_0$, called prewavelets, such that the shifts $\{\psi_\nu(2^j \cdot - \mathbf{m})$, $\mathbf{m} \in \mathbf{Z}^n$, $\nu \in \mathcal{V}'\}$ form an L_2-stable basis in the space W_j [3, 17]. Here we index the prewavelets ψ_ν by the set $\mathcal{V}' = \mathcal{V} \setminus \{\mathbf{0}\}$ with $\mathcal{V}$ denoting the set of vertices of the cube $[0, 1/2]^d$. Thus one obtains an L_2-stable basis of the space V_n consisting of

$$\{\phi(\cdot - \mathbf{m}),\ \mathbf{m} \in \mathbf{Z}^d\} \qquad \text{and} \qquad \{\psi_\nu(2^j \cdot - \mathbf{m}),\ \mathbf{m} \in \mathbf{Z}^d,\ \nu \in \mathcal{V}',\ j = 0, \ldots, n-1\}.$$

Similar to other transform methods elements of V_n and operators are now expanded into the new basis and the computations take place in this system of coordinates, where one hopes to achieve that the computation is faster than in the original system. Additionally, some features of wavelets, such as the localization in both space and frequency domains and vanishing moment properties, lead to a number of new and interesting properties of wavelet-based numerical methods. The multiresolution structure of the wavelet expansion leads to an effective organization of transformations. Furthermore, the vanishing moments of wavelets imply that within a prescribed accuracy pseudodifferential operators admit sparse matrix representations, which allows one to design fast numerical algorithms for these operators. There exist a series of papers on the application of wavelet methods to the computation of integral operators and the solution of integral equations, where different types of scaling functions and wavelets are used (see [1, 2, 8, 18] and the references therein). Since the scaling function ϕ has to satisfy the so-called refinement equation,

$$\phi(\mathbf{x}) = \sum_{\mathbf{m}\in\mathbf{Z}^n} a_{\mathbf{m}}\phi(2\mathbf{x} - \mathbf{m}), \tag{1.4}$$

as a rule in these applications the functions are piecewise polynomials satisfying some smoothness and vanishing moment conditions. Many interesting examples can be found in [6, 9, 17] and the above-mentioned papers. However, one drawback of these functions is that it is practically impossible to derive analytic formulas for the action of important integral operators of mathematical physics on these functions, especially in the multidimensional case. Thus it is necessary to use cubatures for integral operators with singular kernel functions applied to piecewise polynomials.

A different point of view regarding the cubature of integral operators of mathematical physics

$$Ku(\mathbf{x}) = \int_{\mathbf{R}^d} k(\mathbf{x}, \mathbf{y})u(\mathbf{y})d\mathbf{y}$$

was developed in [12–14, 16], where the function u is approximated by linear combinations of the form

$$u_h(\mathbf{x}) := \mathcal{D}^{-n/2} \sum_{\mathbf{m}\in\mathbf{Z}^d} u_{\mathbf{m}}\eta\left(\frac{\mathbf{x} - h\mathbf{m}}{\sqrt{\mathcal{D}}h}\right). \tag{1.5}$$

Here the generating function η decays together with its Fourier transform $\mathcal{F}\eta$ rapidly at infinity and has the property that the integral $K\eta$ can be evaluated very effectively, either analytically or by simple one-dimensional quadrature. For example, the Newton potential of the function $\eta(\mathbf{x}) = (\frac{5}{2} - |\mathbf{x}|^2)\exp(-|\mathbf{x}|^2)$, $\mathbf{x} \in \mathbf{R}^3$, is given by

$$\mathcal{N}\eta(\mathbf{x}) := \frac{1}{4\pi}\int_{\mathbf{R}^3} \frac{(5/2 - |\mathbf{y}|^2)e^{-|\mathbf{y}|^2}}{|\mathbf{x} - \mathbf{y}|}d\mathbf{y} = \frac{1}{2\pi^{3/2}}\left(\frac{1}{|\mathbf{x}|}\int_0^{|\mathbf{x}|} e^{-\tau^2}d\tau + e^{-|\mathbf{x}|^2}\right). \tag{1.6}$$

Hence, if the density can be represented as the sum $u_h(\mathbf{x})$ one obtains analytic formulas for the Newton potential and its derivatives, for example.

The crucial problem of this approach lies in the approximation properties of the spaces $\{\eta((\mathbf{x} - h\mathbf{m})/\sqrt{\mathcal{D}}h), \mathbf{m} \in \mathbf{Z}^d\}$, where the parameter $\mathcal{D} > 0$ is fixed and $h \to 0$. In [15] we proved that if the generating function η satisfies the moment conditions

$$\int_{\mathbf{R}^d} \eta(\mathbf{x})d\mathbf{x} = 1, \qquad \int_{\mathbf{R}^d} \mathbf{x}^{\boldsymbol{\alpha}}\eta(\mathbf{x})d\mathbf{x} = 0,\ \forall\boldsymbol{\alpha},\ 1 \le |\boldsymbol{\alpha}| < N \tag{1.7}$$

(here we use standard multi-index notations), then the quasi-interpolant,

$$u_h(\mathbf{x}) := \mathcal{D}^{-d/2} \sum_{\mathbf{m}\in\mathbf{Z}^d} u(h\mathbf{m})\eta\left(\frac{\mathbf{x}-h\mathbf{m}}{\sqrt{\mathcal{D}}h}\right), \tag{1.8}$$

provides the following representation: For any $u \in C^N(\mathbf{R}^d) \cap W_\infty^N(\mathbf{R}^d)$ there holds

$$u_h(\mathbf{x}) = u(\mathbf{x}) + R_h(\mathbf{x}), \tag{1.9}$$

where

$$|R_h(\mathbf{x})| \le c_\eta(\sqrt{\mathcal{D}}h)^N \|\nabla_N u\|_{L_\infty(\mathbf{R}^d)} + |u(\mathbf{x})| O\Big(\sum_{\boldsymbol{\nu}\in\mathbf{Z}^d\setminus\{\mathbf{0}\}} |\mathcal{F}\eta(\sqrt{\mathcal{D}}\boldsymbol{\nu})|\Big). \tag{1.10}$$

Due to the second term the quasi-interpolant u_h does not converge to u. However, the rapid decay of $\mathcal{F}\eta$ ensures that one can choose $\mathcal{D}$ such that this saturation error can be made arbitrarily small, for example, less than the needed accuracy or the machine precision. Then u_h behaves in numerical computations like a usual approximant of the order $O(h^N)$. This behavior was studied in the framework of "approximate approximations" in [13–15], where the approximation error was estimated in different norms, and the construction of suitable generating functions for quasi-interpolation formulas and other simple approximants were described.

Applied to the example (1.6) concerning the Newton potential this approach leads to the following result:

If $u \in W_p^6(\mathbf{R}^3)$, $1 < p < \frac{3}{2}$, then the cubature formula

$$\mathcal{N}_h u = \frac{h^2}{\sqrt{\mathcal{D}}} \sum_{\mathbf{m}\in\mathbf{Z}^3} u(h\mathbf{m})\mathcal{N}\eta\left(\frac{\mathbf{x}-h\mathbf{m}}{\sqrt{\mathcal{D}}h}\right) \tag{1.11}$$

provides the estimate

$$\|\mathcal{N}u - \mathcal{N}_h u\|_{L_q(\mathbf{R}^3)} \le (c_1(\sqrt{\mathcal{D}}h)^4 + c_2 h^2 e^{-\pi^2\mathcal{D}})\|u\|_{W_p^6(\mathbf{R}^3)} \tag{1.12}$$

with positive constants c_1, c_2, and $q = 3p/(3-2p)$ (for a proof see [16], where also cubature formulas of arbitrary order for different potentials are given). In view of $\exp(-\pi^2) = 0.51723\ldots \times 10^{-4}$ we see that for sufficiently large $\mathcal{D}$, say $\mathcal{D} \ge 4$, formula (1.11) behaves in numerical computations like a fourth-order cubature. Thus giving up the requirement of convergence of the approximants (1.5) as $h \to 0$ does not imply any serious restrictions, as far as numerical computations are concerned and η and $\mathcal{D}$ are suitable chosen. But one obtains an essentially greater flexibility in the choice of basic functions resulting in simple multidimensional approximation formulas and cubatures of important classes of integral operators. In many cases these formulas are semi-analytic such that other analytic operations like, for example, taking the gradients of potentials, often required in solving applied problems, can be carried out efficiently and with high accuracy (cf. [13, 16] for other examples and applications).

It is clear that the basic functions we have in mind do not satisfy a refinement equation of the form (1.4). But it turns out that for a wide class of interesting functions refinement equations are valid in some approximate sense. For example, for $\mathbf{x} \in \mathbf{R}^d$ we have

$$\left| e^{-|\mathbf{x}|^2/\mathcal{D}} - \frac{2^d}{(3\pi\mathcal{D})^{d/2}} \sum_{\mathbf{m}\in\mathbf{Z}^d} e^{-|\mathbf{m}|^2/3\mathcal{D}} e^{-|2\mathbf{x}-\mathbf{m}|^2/\mathcal{D}} \right| \leq (2+\epsilon) d e^{-3\pi^2\mathcal{D}/4} e^{-|\mathbf{x}|^2/\mathcal{D}}$$

with $\epsilon \ll 1$, such that the Gaussian function $\phi_{\mathcal{D}}(\mathbf{x}) := e^{-|\mathbf{x}|^2/\mathcal{D}}$ satisfies a refinement equation within any prescribed tolerance if $\mathcal{D}$ is chosen sufficiently large. This leads to the idea of performing an approximate multiresolution analysis and wavelet construction similar to the case where one has an exact refinement equation. This is the goal of our paper.

In Section 2 we prove that there exists a large class of basic functions satisfying approximate refinement equations and study some approximation properties of the Gaussian radial function $\phi_{\mathcal{D}}$. For this example we provide in Section 3 an approximate multiresolution analysis. We show that any element of the L_2-closure of the linear span

$$\mathbf{V}_n := \{\phi_{\mathcal{D}}(2^n \cdot -\mathbf{m}),\ \mathbf{m} \in \mathbf{Z}^d\} \tag{1.13}$$

can be approximated by elements of the direct sum

$$\tilde{\mathbf{V}}_n := \mathbf{V}_0 \dot{+} \mathbf{W}_0 \dot{+} \cdots \dot{+} \mathbf{W}_{n-1}, \tag{1.14}$$

with some small relative error of the form $\varepsilon = (4+\epsilon) dn e^{-3\pi^2\mathcal{D}/4}$, $\epsilon \ll 1$. Here the wavelet spaces $\mathbf{W}_j$ are almost orthogonal such that the approximate decomposition (1.14) of $\mathbf{V}_n$ can be performed using the orthogonal projections P_0 onto $\mathbf{V}_0$ and Q_j onto $\mathbf{W}_j$.

The univariate wavelet construction and some properties of the prewavelet are discussed in Section 4. The univariate wavelet spaces W_j are spanned by rapidly decaying analytic prewavelets, plotted in Fig. 1, which belong to V_{j+1} and are orthogonal to all elements of V_j.

It is interesting that there exist simple analytic formulas for a perturbation of the wavelet with a relative error ε^2. More precisely, the space W_0 can be defined within the assumed tolerance ε as the span of the integer translates of the function

$$\psi_{\mathcal{D}}(x) = e^{-(2x-1)^2/6\mathcal{D}} \cos \frac{5\pi}{6}(2x-1). \tag{1.15}$$

Since the values of many integral operators applied to the wavelets can be given analytically one can use approximants from $\tilde{\mathbf{V}}_n$ to derive the cubature of these operators. Here one assumes that the transformation to the basis in $\tilde{\mathbf{V}}_n$ leads to data compression, at least for sufficiently smooth integrands u. Additionally, the moments of the prewavelets

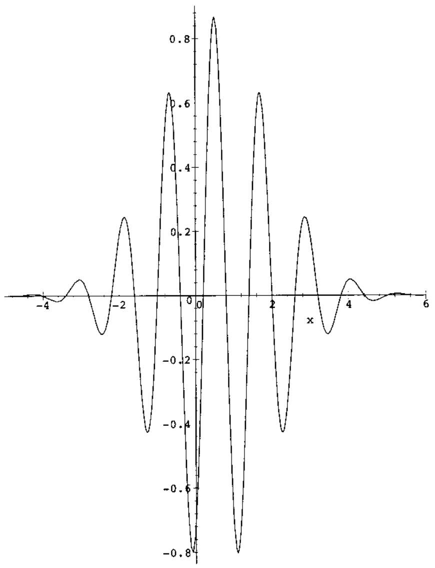

FIG. 1. Graph of the wavelet function $\psi_3(x)$.

are very small and can be controlled by the parameter $\mathcal{D}$. This implies, for example, a fast decay of the integrals $K\psi$ if the kernel $k(\mathbf{x}, \mathbf{y})$ satisfies

$$\left|\partial_y^{\boldsymbol{\alpha}} k(\mathbf{x}, \mathbf{y})\right| \leq c_{\boldsymbol{\alpha}} \left|\mathbf{x} - \mathbf{y}\right|^{-(\gamma + |\boldsymbol{\alpha}|)} \qquad \text{for some } \gamma > 0.$$

The effect of the nearly vanishing moments is shown in Fig. 2 for the example of the Hilbert transform of $\psi_{\mathcal{D}}$,

$$\begin{aligned}\mathcal{H}\psi_{\mathcal{D}}(x) &:= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\psi_{\mathcal{D}}(y)}{y - x}\, dy \\ &= \frac{ie^{-25\pi^2\mathcal{D}/24}}{4} \left(W\!\left(\frac{5\pi\mathcal{D} - 2i(2x-1)}{2\sqrt{6\mathcal{D}}} \right) - W\!\left(\frac{5\pi\mathcal{D} + 2i(2x-1)}{2\sqrt{6\mathcal{D}}} \right) \right),\end{aligned}$$

where we use the complementary error function erfc to define the function W as

$$W(z) := e^{z^2}(\operatorname{erfc}(z) - \operatorname{erfc}(-z)).$$

We see that the essential supports of $\psi_{\mathcal{D}}$ and $\mathcal{H}\psi_{\mathcal{D}}$ are very close, which leads to a high compression rate for the matrix representation of the Hilbert transform in the basis

$$\{\phi_{\mathcal{D}}(\cdot - k),\, k \in \mathbf{Z}\} \qquad \text{and} \qquad \{\psi_{\mathcal{D}}(2^j \cdot - k),\, k \in \mathbf{Z},\, j = 0, \ldots, n-1\}.$$

In Section 5 we consider multivariate approximate wavelets and the multiresolution

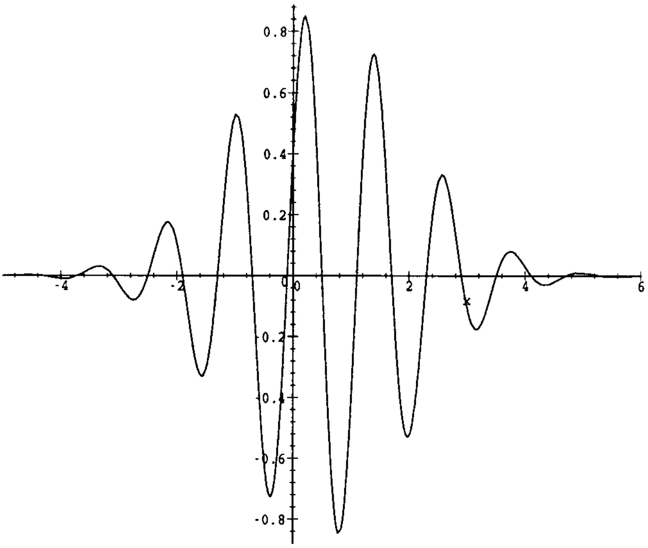

FIG. 2. Graph of the Hilbert transform.

structure of the spaces spanned by the Gaussian radial function. We construct a wavelet basis with the property that important pseudodifferential operators admit semi-analytic representations. Further we give explicit formulas for the orthogonal projection P_0 onto $\mathbf{V}_0$ and almost orthogonal projections $\tilde{Q}_j$ onto decomposition (1.14), $\mathbf{W}_j$, which are proved in the final section, Section 6, such that for any $\varphi_n \in \mathbf{V}_n$ the estimate

$$\|\varphi_n - P_0\varphi_n - \sum_{j=0}^{n-1} \tilde{Q}_j\varphi_n\|_2 \leq c\varepsilon\|\varphi_n\|_2 \tag{1.16}$$

holds with some constant not depending on φ_n and $\mathcal{D}$.

Consequently, the proposed approximate multiresolution analysis combines the advantages of well-established wavelet methods in numerical analysis with the use of simple approximating formulas based on smooth generating functions. The drawback of nonconvergence and nonexact refinement equations can be overcome by an appropriate choice of parameters to force the errors within the round-off required. By this construction, which seems to be new, it is possible to obtain functions with properties which are difficult or impossible to achieve by considering exact stationary multiresolution analysis. The approximate wavelets combine excellent space and frequency localization with a high number of nearly vanishing moments, so it would also be interesting to use them in other applications of wavelet functions.

Let us remark that there exists a large bibliography concerning the application of the Gaussian and related functions in approximation and wavelet theory. Already the Gabor wavelet proposed in 1946 represents a modulated Gaussian; other known examples are the Morlet wavelet and the Mexican hat functions (cf. [5, 9]). Note that the wavelet function (1.15) is the real part of some special Gabor wavelet. Recently the adaptive signal representation using dictionaries of scaled, modulated, and translated Gabor functions was studied in a series of papers by Mallat and coauthors (cf. [10] and the references therein). In [7], Chui *et al.* use nonstationary multiresolution analysis to construct analytic wavelets of radial basis functions, in particular for the Gaussian scaling function. Here the spaces V_j are spanned by the shifts $\phi(\cdot - 2^{-j}\mathbf{m})$, $\mathbf{m} \in \mathbf{Z}^d$, of the scaling function ϕ. In this case the spaces V_j are nested, and standard wavelet constructions can be applied. However, these analytic wavelets are not localized and are different on different scales.

2. APPROXIMATE REFINEMENT EQUATIONS

Suppose that $u(\mathbf{x})$, $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbf{R}^d$, is the restriction of an entire function $u(\mathbf{z})$, $\mathbf{z} = (z_1, \ldots, z_d) \in \mathbf{C}^d$, such that for some fixed positive parameters h and $\mathcal{D}$ and any $\mathbf{x} \in \mathbf{R}^d$ the estimate

$$\left| u(\mathbf{x} + i\pi\mathcal{D}h\mathbf{y})\exp(-\pi^2\mathcal{D}|\mathbf{y}|^2)\right| \le A(1 + |\mathbf{y}|)^{-d-\delta}, \qquad \mathbf{y} \in \mathbf{R}^d, \tag{2.1}$$

holds. Then for any $\mathbf{m} \in \mathbf{Z}^d$ there exists the number

$$u_{\mathbf{m}} := \int_{\mathbf{R}^d} \exp(-\pi^2\mathcal{D}|\mathbf{y}|^2)u(h\mathbf{m} + i\pi\mathcal{D}h\mathbf{y})\,d\mathbf{y}. \tag{2.2}$$

In the following we denote by $\langle \mathbf{x}, \mathbf{y}\rangle$ the usual scalar product in $\mathbf{R}^d$, $|\mathbf{x}| = \langle \mathbf{x}, \mathbf{x}\rangle^{1/2}$, and the Fourier transform is defined as

$$\mathcal{F}\varphi(\boldsymbol{\lambda}) = \int_{\mathbf{R}^d} \varphi(\mathbf{x})e_{\boldsymbol{\lambda}}(-\mathbf{x})\,d\mathbf{x} \qquad \text{with } e_{\boldsymbol{\lambda}}(\mathbf{x}) := e^{2\pi i\langle \mathbf{x}, \boldsymbol{\lambda}\rangle}.$$

LEMMA 2.1. *If the entire function u satisfies* (2.1) *and for a given $\mathbf{x} \in \mathbf{R}^d$ the semi-discrete convolution*

$$u_h(\mathbf{x}) := \sum_{\mathbf{m}\in\mathbf{Z}^d} u_{\mathbf{m}}\exp\left(-\frac{|\mathbf{x} - h\mathbf{m}|^2}{\mathcal{D}h^2}\right) = \sum_{\mathbf{m}\in\mathbf{Z}^d} u_{\mathbf{m}}\phi_{\mathcal{D}}(\mathbf{x}/h - \mathbf{m})$$

converges absolutely, then the equality

$$u_h(\mathbf{x}) = \sum_{\boldsymbol{\nu}\in\mathbf{Z}^d} u(\mathbf{x} + i\pi\mathcal{D}h\boldsymbol{\nu})\exp(-\pi^2\mathcal{D}|\boldsymbol{\nu}|^2)e_{\boldsymbol{\nu}}(\mathbf{x}/h)$$

holds.

Proof. By repeated application of Cauchy's Theorem one gets

$$\begin{aligned}
u_{\mathbf{m}}\phi_{\mathcal{D}}(\mathbf{x}/h-\mathbf{m}) &= (\pi\mathcal{D}h)^{-n}\phi_{\mathcal{D}}(\mathbf{x}/h-\mathbf{m})\int_{\mathbf{R}^d} u(h\mathbf{m}+i\mathbf{y})\phi_{\mathcal{D}}(\mathbf{y}/h)d\mathbf{y} \\
&= (\pi\mathcal{D}h)^{-n}\phi_{\mathcal{D}}(\mathbf{x}/h)\int_{\mathbf{R}^d} u(i\mathbf{y})\phi_{\mathcal{D}}(\mathbf{y}/h)e_{\mathbf{m}}\left(-\frac{2i}{\mathcal{D}h}(\mathbf{y}+i\mathbf{x})\right)d\mathbf{y} \\
&= (\pi\mathcal{D}h)^{-n}\int_{\mathbf{R}^d} u(\mathbf{x}+i\mathbf{y})\phi_{\mathcal{D}}(\mathbf{y}/h)e_{\mathbf{y}}\left(\frac{2i}{\mathcal{D}h}(\mathbf{x}/h-\mathbf{m})\right)d\mathbf{y} \\
&= \int_{\mathbf{R}^d} u(\mathbf{x}+i\pi\mathcal{D}h\mathbf{y})\exp(-\pi^2\mathcal{D}|\mathbf{y}|^2)e_{\mathbf{y}}(\mathbf{x}/h)e_{\mathbf{m}}(-\mathbf{y})d\mathbf{y}.
\end{aligned}$$

Denoting by

$$f_{\mathbf{x}}(\mathbf{y}) := u(\mathbf{x}+i\pi\mathcal{D}h\mathbf{y})\exp(-\pi^2\mathcal{D}|\mathbf{y}|^2)e_{\mathbf{y}}(\mathbf{x}/h)$$

we see that

$$u_{\mathbf{m}}\phi_{\mathcal{D}}(\mathbf{x}/h-\mathbf{m}) = \mathcal{F}f_{\mathbf{x}}(\mathbf{m}).$$

Now we have only to apply Poisson's summation formula (see [19, Thm. VII. 2.4]),

$$\sum_{\mathbf{m}\in\mathbf{Z}^d}\mathcal{F}f_{\mathbf{x}}(\mathbf{m}) = \sum_{\boldsymbol{\nu}\in\mathbf{Z}^d} f_{\mathbf{x}}(\boldsymbol{\nu}) = \sum_{\boldsymbol{\nu}\in\mathbf{Z}^d} u(\mathbf{x}+i\pi\mathcal{D}h\boldsymbol{\nu})\exp(-\pi^2\mathcal{D}|\boldsymbol{\nu}|^2)e_{\boldsymbol{\nu}}(\mathbf{x}/h),$$

which is valid due to the absolute convergence of both series. ■

Thus one obtains the relation

$$u_h(\mathbf{x}) - u(\mathbf{x}) = \sum_{\boldsymbol{\nu}\in\mathbf{Z}^d\setminus\{\mathbf{0}\}} u(\mathbf{x}+i\pi\mathcal{D}h\boldsymbol{\nu})\exp(-\pi^2\mathcal{D}|\boldsymbol{\nu}|^2)e_{\boldsymbol{\nu}}(\mathbf{x}/h), \tag{2.3}$$

showing that if $\mathcal{D}$ is suitable chosen then u_h is for relative large h a very precise approximant to analytic functions u of first order of growth. In this case u has a compactly supported Fourier transform and we obtain an equivalent formula for the coefficients

$$u_{\mathbf{m}} = (\pi\mathcal{D})^{-d/2}\int_{\mathbf{R}^d}\mathcal{F}u(\boldsymbol{\lambda})\exp(\pi^2\mathcal{D}h^2|\boldsymbol{\lambda}|^2)e_{\mathbf{m}}(h\boldsymbol{\lambda})d\boldsymbol{\lambda}.$$

Let us give some examples. First we consider the case of a polynomial u. It can be easily seen that

$$\int_{-\infty}^{\infty} (hm + i\pi\mathcal{D}hy)^j \exp(-\pi^2\mathcal{D}y^2)dy = \frac{1}{\sqrt{\pi\mathcal{D}}}\left(\frac{\sqrt{\mathcal{D}}h}{2}\right)^j H_j\left(\frac{m}{\sqrt{\mathcal{D}}}\right),$$

with the Hermite polynomials

$$H_j(y) := (-1)^j e^{y^2}\left(\frac{d}{dy}\right)^j e^{-y^2}.$$

In particular we obtain from Lemma 2.1 the estimate

$$\left| x^j - \frac{1}{\sqrt{\pi\mathcal{D}}}\left(\frac{\sqrt{\mathcal{D}}}{2}\right)^j \sum_{m=-\infty}^{\infty} H_j\left(\frac{m}{\sqrt{\mathcal{D}}}\right) e^{-(x-m)^2/\mathcal{D}} \right| \le \sum_{k\in\mathbf{Z}\setminus\{0\}} |x + i\pi\mathcal{D}k|^j e^{-\pi^2\mathcal{D}k^2}.$$

Hence, any polynomial $p(\mathbf{x})$ can be approximated by linear combinations of the functions $\phi_{\mathcal{D}}(\mathbf{x} - \mathbf{m})$, $\mathbf{m} \in \mathbf{Z}^d$, with an arbitrary relative error $\epsilon > 0$ if $\mathcal{D}$ is chosen sufficiently large.

Next we consider the example of the exponential function $u(\mathbf{x}) = e^{\langle \mathbf{x},\mathbf{a}\rangle}$, $\mathbf{a} \in \mathbf{C}^d$. Here Lemma 2.1 leads to

$$\begin{aligned} u_h(\mathbf{x}) &= (\pi\mathcal{D})^{-d/2} e^{-\mathcal{D}h^2\mathbf{a}^2/4} \sum_{\mathbf{m}\in\mathbf{Z}^d} e^{\langle h\mathbf{m},\mathbf{a}\rangle}\phi_{\mathcal{D}}(\mathbf{x}/h - \mathbf{m}) \\ &= e^{\langle \mathbf{x},\mathbf{a}\rangle}\left(1 + \sum_{\boldsymbol{\nu}\in\mathbf{Z}^d\setminus\{\mathbf{0}\}} e^{-\pi^2\mathcal{D}|\boldsymbol{\nu}|^2} e_{\boldsymbol{\nu}}\left(\frac{\mathbf{x}}{h} + \frac{\mathcal{D}h\mathbf{a}}{2}\right)\right). \end{aligned}$$

Thus, if $\mathbf{a} \in \mathbf{R}^d$ then for any $h > 0$ the series u_h approximates the exponential function with the relative error

$$\sum_{\boldsymbol{\nu}\in\mathbf{Z}^d\setminus\{\mathbf{0}\}} e^{-\pi^2\mathcal{D}|\boldsymbol{\nu}|^2}.$$

If $\mathbf{a} = \mathbf{u} + i\mathbf{v}$, $\mathbf{u}, \mathbf{v} \in \mathbf{R}^d$ with $\mathbf{v} \neq \mathbf{0}$, and $h \le \pi/2|\mathbf{v}|$ then u_h approximates $e^{\langle \mathbf{x},\mathbf{a}\rangle}$ with the relative error less than or equal to

$$\sum_{\boldsymbol{\nu}\in\mathbf{Z}^d\setminus\{\mathbf{0}\}} e^{-\pi^2\mathcal{D}|\boldsymbol{\nu}|^2/2}.$$

If $d = 1$ then we get some special cases of the well-known transformation formula for Jacobi's elliptic theta functions

$$\sum_{k\in\mathbf{Z}} e^{-k^2\pi a + 2\pi i k z} = a^{-1/2}\sum_{m\in\mathbf{Z}} e^{-\pi(z-m)^2/a}, \qquad z \in \mathbf{C}, \operatorname{Re} a > 0, \tag{2.4}$$

in particular

$$\cos 2\pi x = \frac{e^{\pi^2\mathcal{D}/4}}{2\sqrt{\pi\mathcal{D}}} \sum_{m=-\infty}^{\infty} (-1)^m e^{-(2x-m)^2/\mathcal{D}} - R(x)$$

$$\text{where } R(x) = e^{\pi^2\mathcal{D}/4} \sum_{k=1}^{\infty} \cos(2\pi(2k+1)x) e^{-\pi^2\mathcal{D}(2k+1)^2/4}$$

$$= O(e^{-2\pi^2\mathcal{D}}). \tag{2.5}$$

Finally we apply Lemma 2.1 to $\varphi_{\mathcal{D}}(\mathbf{x})$. If $h < 1$ then we obtain the equality

$$\varphi_{\mathcal{D}}(\mathbf{x}) = (\pi\mathcal{D}(1-h^2))^{-d/2} \sum_{\mathbf{m}\in\mathbf{Z}^d} e^{-h^2|\mathbf{m}|^2/\mathcal{D}(1-h^2)} \varphi_{\mathcal{D}}(\mathbf{x}/h - \mathbf{m})$$
$$- \varphi_{\mathcal{D}}(\mathbf{x}) \sum_{\boldsymbol{\nu}\in\mathbf{Z}^d\setminus\{\mathbf{0}\}} e_{\boldsymbol{\nu}/h}((1-h^2)\mathbf{x}) e^{-\pi^2\mathcal{D}(1-h^2)|\boldsymbol{\nu}|^2}. \tag{2.6}$$

Therefore the Gaussian function satisfies the approximate refinement equation

$$\varphi_{\mathcal{D}}(\mathbf{x}) \doteq (\pi\mathcal{D}(1-h^2))^{-d/2} \sum_{\mathbf{m}\in\mathbf{Z}^d} e^{-h^2|\mathbf{m}|^2/\mathcal{D}(1-h^2)} \varphi_{\mathcal{D}}(\mathbf{x}/h - \mathbf{m}), \tag{2.7}$$

with the accuracy $\varphi_{\mathcal{D}}(\mathbf{x}) \sum_{\boldsymbol{\nu}\in\mathbf{Z}^d\setminus\{\mathbf{0}\}} e^{-\pi^2\mathcal{D}(1-h^2)|\boldsymbol{\nu}|^2}$.

Approximate refinement equations of the from (2.7) are valid for a large class of basic functions η as shown by the following assertion.

LEMMA 2.2. *Let η belong to the Schwartz space, $\eta \in \mathcal{S}(\mathbf{R}^d)$, with positive Fourier transform $\mathcal{F}\eta > 0$ and such that for given $h < 1$ the function $g \in \mathcal{S}(\mathbf{R}^d)$, which is defined via its Fourier transform by $\mathcal{F}g(\boldsymbol{\lambda}) := \mathcal{F}\eta(\boldsymbol{\lambda})/\mathcal{F}\eta(h\boldsymbol{\lambda})$. Then for any $\varepsilon > 0$ there exists $\mathcal{D} > 0$ such that for all $\mathbf{x} \in \mathbf{R}^n$,*

$$\left| \eta\left(\frac{\mathbf{x}}{\sqrt{\mathcal{D}}}\right) - \mathcal{D}^{-d/2} \sum_{\mathbf{m}\in\mathbf{Z}^d} g\left(\frac{h\mathbf{m}}{\sqrt{\mathcal{D}}}\right) \eta\left(\frac{\mathbf{x} - h\mathbf{m}}{\sqrt{\mathcal{D}}h}\right) \right| < \varepsilon. \tag{2.8}$$

Proof. Again we may apply the Poisson summation formula to

$$\mathcal{D}^{-d/2} \sum_{\mathbf{m}\in\mathbf{Z}^d} g\left(\frac{h\mathbf{m}}{\sqrt{\mathcal{D}}}\right) \eta\left(\frac{\mathbf{x} - h\mathbf{m}}{\sqrt{\mathcal{D}}h}\right) = \sum_{\boldsymbol{\nu}\in\mathbf{Z}^d} \int_{\mathbf{R}^d} g(h\mathbf{y}) \eta\left(\frac{\mathbf{x}}{\sqrt{\mathcal{D}}h} - \mathbf{y}\right) e_{\boldsymbol{\nu}}(\sqrt{\mathcal{D}}\mathbf{y}) d\mathbf{y}$$

$$= \sum_{\boldsymbol{\nu}\in\mathbf{Z}^d} \int_{\mathbf{R}^d} \mathcal{F}g(\boldsymbol{\lambda}) \int_{\mathbf{R}^d} \eta(\mathbf{y}) e_{h\boldsymbol{\lambda}+\sqrt{\mathcal{D}}\boldsymbol{\nu}}\left(\frac{\mathbf{x}}{\sqrt{\mathcal{D}}h} - \mathbf{y}\right) d\mathbf{y}\, d\boldsymbol{\lambda}$$

$$= \sum_{\boldsymbol{\nu}\in\mathbf{Z}^d} e_{\boldsymbol{\nu}}\left(\frac{x}{h}\right) \int_{\mathbf{R}^d} \frac{\mathcal{F}\eta(\boldsymbol{\lambda})}{\mathcal{F}\eta(h\boldsymbol{\lambda})} \mathcal{F}\eta(h\boldsymbol{\lambda} + \sqrt{\mathcal{D}}\boldsymbol{\nu}) e_{\boldsymbol{\lambda}}\left(\frac{\mathbf{x}}{\sqrt{\mathcal{D}}}\right) d\boldsymbol{\lambda}$$

$$= \eta\left(\frac{\mathbf{x}}{\sqrt{\mathcal{D}}}\right) + \sum_{\boldsymbol{\nu}\in\mathbf{Z}^d\setminus\{\mathbf{0}\}} e_{\boldsymbol{\nu}}\left(\frac{\mathbf{x}}{h}\right) \int_{\mathbf{R}^d} \frac{\mathcal{F}\eta(\boldsymbol{\lambda})}{\mathcal{F}\eta(h\boldsymbol{\lambda})} \mathcal{F}\eta(h\boldsymbol{\lambda} + \sqrt{\mathcal{D}}\boldsymbol{\nu}) e_{\boldsymbol{\lambda}}\left(\frac{\mathbf{x}}{\sqrt{\mathcal{D}}}\right) d\boldsymbol{\lambda}.$$

Now it is easy to see that the function of $\mathbf{y} \in \mathbf{R}^d$,

$$\int_{\mathbf{R}^d} \frac{\mathcal{F}\eta(\boldsymbol{\lambda})}{\mathcal{F}\eta(h\boldsymbol{\lambda})} \mathcal{F}\eta(h\boldsymbol{\lambda} + \mathbf{y})\, d\boldsymbol{\lambda} \in \mathcal{S}(\mathbf{R}^d),$$

and hence the sum

$$\sum_{\boldsymbol{\nu} \in \mathbf{Z}^d \setminus \{\mathbf{0}\}} \left| \int_{\mathbf{R}^d} \frac{\mathcal{F}\eta(\boldsymbol{\lambda})}{\mathcal{F}\eta(h\boldsymbol{\lambda})} \mathcal{F}\eta(h\boldsymbol{\lambda} + \sqrt{\mathcal{D}}\boldsymbol{\nu}) e_{\boldsymbol{\lambda}}\left(\frac{\mathbf{x}}{\sqrt{\mathcal{D}}}\right) d\boldsymbol{\lambda} \right|,$$

can be made arbitrarily small by choosing $\mathcal{D}$ large enough. ∎

We note a special case of basic functions giving high-order semi-analytic quasi-interpolation formulas of the form (1.8). Using the generalized Laguerre polynomials $L_j^{(\alpha)}$ we define the basic functions (see [13, 14])

$$\eta_{2M}(\mathbf{x}) := \pi^{-d/2} L_{M-1}^{(d/2)}(|\mathbf{x}|^2) e^{-|\mathbf{x}|^2}, \qquad M = 1, 2, \ldots,$$

having the Fourier transform

$$\mathcal{F}\eta_{2M}(\boldsymbol{\lambda}) = e^{-\pi^2|\boldsymbol{\lambda}|^2} \sum_{j=0}^{M-1} \frac{(\pi^2|\boldsymbol{\lambda}|^2)^j}{j!}$$

and thus satisfying the moment conditions (1.7) with $N = 2M$. Note that the basic function η, considered in (1.6), corresponds to the case $N = 4$ and $d = 3$. For the functions η_{2M} one can estimate the error bound of Lemma 2.2 by

$$\varepsilon \leq \sum_{\boldsymbol{\nu} \in \mathbf{Z}^d \setminus \{\mathbf{0}\}} q(\sqrt{\mathcal{D}}\boldsymbol{\nu}) e^{-\pi^2 \mathcal{D}(1-h^2)|\boldsymbol{\nu}|^2}$$

with some polynomial q of degree $2M$.

It is clear that the mask values $g(h\mathbf{m}/\sqrt{\mathcal{D}})$ in (2.8) can be computed by simple one-dimensional quadratures if η is the radial function. Some of the above-mentioned basic functions allow one to derive analytic formulas for the function g. For example, for the function η considered in (1.6) one has

$$g(\mathbf{x}) = \frac{e^{-|\mathbf{x}|^2/(1-h^2)}}{h^2\sqrt{\pi\mathcal{D}(1-h^2)}^3} - \frac{1-h^2}{h^4} \frac{e^{(1-h^2)/h^2}}{2\pi|\mathbf{x}|\sqrt{\mathcal{D}}^3} \times \left(e^{-2|\mathbf{x}|/h} \operatorname{erfc}\left(\frac{1 - h|\mathbf{x}| - h^2}{h\sqrt{1-h^2}}\right) - e^{2|\mathbf{x}|/h} \operatorname{erfc}\left(\frac{1 + h|\mathbf{x}| - h^2}{h\sqrt{1-h^2}}\right)\right).$$

We remark that the knowledge of g is very useful also for another interesting application of the approximate refinement equation, which allows one to derive quasi-interpolation formulas for nonuniformly distributed mesh points.

3. APPROXIMATE MULTIRESOLUTION

In this section we provide for the example of the Gaussian function $\phi_{\mathcal{D}}$ an approximate multiresolution analysis. We introduce the closed linear subspaces of $L_2(\mathbf{R}^d)$

$$\mathbf{V}_j := \{ \sum_{\mathbf{m}\in\mathbf{Z}^d} a_{\mathbf{m}} \phi_{\mathcal{D}}(2^j \cdot\, - \mathbf{m}),\ \{a_{\mathbf{m}}\} \in l_2(\mathbf{Z}^d)\}.$$

Since

$$\begin{aligned} \| \sum_{\mathbf{m}\in\mathbf{Z}^d} a_{\mathbf{m}} \phi_{\mathcal{D}}(\cdot\, - \mathbf{m})\|_2^2 &= (\pi\mathcal{D})^d \int_{\mathbf{R}^d} e^{-2\pi^2\mathcal{D}|\boldsymbol{\lambda}|^2} \Big| \sum_{\mathbf{m}\in\mathbf{Z}^d} a_{\mathbf{m}} e_{\mathbf{m}}(\boldsymbol{\lambda})\Big|^2 d\boldsymbol{\lambda} \\ &= (\pi\mathcal{D})^d \int_{[0,1]^d} \sum_{\mathbf{k}\in\mathbf{Z}^d} e^{-2\pi^2\mathcal{D}|\boldsymbol{\lambda}-\mathbf{k}|^2} \Big| \sum_{\mathbf{m}\in\mathbf{Z}^d} a_{\mathbf{m}} e_{\mathbf{m}}(\boldsymbol{\lambda})\Big|^2 d\boldsymbol{\lambda}, \end{aligned}$$

the set $\{\phi_{\mathcal{D}}((\cdot - \mathbf{m})\}_{\mathbf{m}\in\mathbf{Z}^d}$ is an L_2-stable basis in $\mathbf{V}_0$. From the approximate refinement equation (2.6) for the Gaussian function it is clear that for any $l < j$ the space $\mathbf{V}_l$ is almost included in $\mathbf{V}_j$. In particular, if $h = \frac{1}{2}$ then we have

$$\phi_{\mathcal{D}}(\mathbf{x}) = \frac{2^d}{(3\pi\mathcal{D})^{d/2}} \sum_{\mathbf{m}\in\mathbf{Z}^d} e^{-|\mathbf{m}|^2/3\mathcal{D}} \phi_{\mathcal{D}}(2\mathbf{x} - \mathbf{m}) - \phi_{\mathcal{D}}(\mathbf{x}) \sum_{\boldsymbol{\nu}\in\mathbf{Z}^d\setminus\{\mathbf{0}\}} e^{-3\pi^2\mathcal{D}|\boldsymbol{\nu}|^2/4} e^{3\pi i\langle x,\boldsymbol{\nu}\rangle};$$

hence for any $\varphi_j \in \mathbf{V}_j$ the small perturbation

$$\varphi_j(\mathbf{x})(1 + \sum_{\boldsymbol{\nu}\in\mathbf{Z}^d\setminus\{\mathbf{0}\}} e^{-3\pi^2\mathcal{D}|\boldsymbol{\nu}|^2/4} e^{3\pi i\langle \mathbf{x},\boldsymbol{\nu}\rangle}) \in \mathbf{V}_{j+1}. \tag{3.1}$$

Consequently, for any $\varphi_j \in \mathbf{V}_j$ we can find $\varphi_{j+1} \in \mathbf{V}_{j+1}$ such that

$$\|\varphi_j - \varphi_{j+1}\|_2 \leq \varepsilon \|\varphi_j\|_2 \tag{3.2}$$

with

$$\varepsilon = \varepsilon_{\mathcal{D}} := \sum_{\boldsymbol{\nu}\in\mathbf{Z}^d\setminus\{\mathbf{0}\}} e^{-3\pi^2\mathcal{D}|\boldsymbol{\nu}|^2/4} = (2 + \epsilon) d e^{-3\pi^2\mathcal{D}/4} \qquad \text{with } \epsilon \ll 1.$$

Furthermore, if we introduce the closed subspace $\mathbf{W}_j \subset \mathbf{V}_{j+1}$ of all functions which are orthogonal to $\mathbf{V}_j$, then it can be easily seen that

$$\big|(\varphi_j, \varphi_l)_2\big| \leq ((1 + \varepsilon)^{|j-l|-1} - 1)\|\varphi_j\|_2 \|\varphi_l\|_2$$

for $\varphi_j \in \mathbf{W}_j$. Thus the situation is very similar to the case when exact refinement equations are valid, which was mentioned in the Introduction.

Let us fix some integer $n > 0$, which determines the grid for the approximating functions. In the following we show that any element of $\mathbf{V}_n$ can be represented within some prescribed tolerance as an element of the multiresolution structure

$$\tilde{\mathbf{V}}_n := \mathbf{V}_0 \dot{+} \mathbf{W}_0 \dot{+} \cdots \dot{+} \mathbf{W}_{n-1}. \tag{3.3}$$

To this end we introduce the orthogonal projections $P_j: L_2(\mathbf{R}^d) \to \mathbf{V}_j$, $j = 0, \ldots, n$, and $Q_j: L_2(\mathbf{R}^d) \to \mathbf{W}_j$, $j = 0, \ldots, n-1$, and denote $Q_{-1} = P_0$.

THEOREM 3.1. *Any $\varphi_n \in \mathbf{V}_n$ can be approximatively represented as an element of the multiresolution structure* (3.3) *and there holds*

$$\|\varphi_n - \sum_{j=-1}^{n-1} Q_j\varphi_n\|_2 \leq n \, \frac{2\varepsilon - \varepsilon^2}{1-\varepsilon} \, \|\varphi_n\|_2.$$

Proof. Use the telescopic series

$$\varphi_n = P_n\varphi_n = \sum_{j=1}^{n} (P_j - P_{j-1})\varphi_n + P_0\varphi_n$$

$$= \sum_{j=1}^{n} (P_j - P_{j-1})\varphi_n + Q_{-1}\varphi_n = \sum_{j=-1}^{n-1} Q_j\varphi_n + \sum_{j=0}^{n-1} (P_{j+1} - P_j - Q_j)\varphi_n$$

and

LEMMA 3.1. *For $j = 0, \ldots, n-1$ we have*

$$\|P_j + Q_j - P_{j+1}\|_2 \leq \frac{2\varepsilon - \varepsilon^2}{1-\varepsilon}.$$

Proof. Note that (3.2) implies the inequality

$$\|\varphi_j - P_{j+1}\varphi_j\|_2 \leq \varepsilon\|\varphi_j\|_2, \qquad \forall \varphi_j \in \mathbf{V}_j. \tag{3.4}$$

Since $P_{j+1}(\mathbf{V}_j) = \mathbf{V}_{j+1} \ominus \mathbf{W}_j$ any $\varphi_{j+1} \in \mathbf{V}_{j+1}$ can be written in the form

$$\varphi_{j+1} = P_{j+1}\varphi_j + Q_j\varphi_{j+1}$$

with some $\varphi_j \in \mathbf{V}_j$. From (3.4) we derive therefore

$$\|\varphi_{j+1} - (\varphi_j + Q_j\varphi_{j+1})\|_2 = \|P_{j+1}\varphi_j - \varphi_j\|_2 \leq \varepsilon\|\varphi_j\|_2$$

and

$$\|\varphi_j\|_2 \le \frac{1}{1-\varepsilon}\,\|P_{j+1}\varphi_j\|_2 = \frac{1}{1-\varepsilon}\,\|\varphi_{j+1} - Q_j\varphi_{j+1}\|_2 \le \frac{\|\varphi_{j+1}\|_2}{1-\varepsilon}.$$

Now we use that the sum $P_j + Q_j$ is the orthogonal projection onto $\mathbf{V}_j \oplus \mathbf{W}_j$. Hence for any $u \in L_2(\mathbf{R}^d)$ we obtain the estimate

$$\|(I - (P_j + Q_j))\,P_{j+1}u\|_2 = \inf_{v \in \mathbf{V}_j \oplus \mathbf{W}_j} \|P_{j+1}u - v\|_2 \le \frac{\varepsilon}{1-\varepsilon}\,\|u\|_2,$$

leading together with (3.4) to

$$\begin{aligned}\|(P_j + Q_j - P_{j+1})u\|_2 &\le \|(I - P_{j+1})(P_j + Q_j)u\|_2 + \|P_{j+1}(I - P_j - Q_j)u\|_2\\ &\le \inf_{v \in \mathbf{V}_{j+1}} \|(P_j + Q_j)u - v\|_2 + \|(I - P_j - Q_j)\,P_{j+1}\|_2\|u\|_2\\ &\le \left(\varepsilon + \frac{\varepsilon}{1-\varepsilon}\right)\|u\|_2. \qquad \blacksquare\end{aligned}$$

4. APPROXIMATE UNIVARIATE WAVELETS

Now we apply some well-known constructions from wavelet theory to the univariate case. Here $\phi_{\mathcal{D}}(x) = e^{-x^2/\mathcal{D}}$; we denote the corresponding scaled and wavelet spaces by V_j and W_j, respectively. Since the function

$$\sum_{m\in\mathbf{Z}} (-1)^{m-1}\mu_{m-1}\phi_{\mathcal{D}}(2\,\cdot\, - m) \qquad \text{with } \mu_m = \int_{\mathbf{R}} \phi_{\mathcal{D}}(x)\phi_{\mathcal{D}}(2x + m)\,dx$$

is orthogonal to all integer shifts of the scaling function $\phi_{\mathcal{D}}$ we obtain a first element of the wavelet space

$$\sum_{m\in\mathbf{Z}} (-1)^{m-1} e^{(m-1)^2/5\mathcal{D}} \phi_{\mathcal{D}}(2x - m) \in W_0.$$

Using (2.5) we derive the formula

$$\begin{aligned}&\sum_{m\in\mathbf{Z}} (-1)^m e^{-m^2/5\mathcal{D}} \phi_{\mathcal{D}}(2x - 1 - m)\\ &\qquad = e^{-(2x-1)^2/6\mathcal{D}} \sum_{m\in\mathbf{Z}} (-1)^m \exp\left(-\frac{6}{5\mathcal{D}}\left(\frac{5}{6}\,(2x-1) - m\right)^2\right)\\ &\qquad = \sqrt{\frac{10\pi\mathcal{D}}{3}}\, e^{-5\pi^2\mathcal{D}/24} e^{-(2x-1)^2/6\mathcal{D}}\left(\cos\frac{5\pi}{6}\,(2x-1) + R_{\mathcal{D}}(x)\right),\end{aligned}$$

with

$$R_{\mathcal{D}}(x) = \sum_{k=1}^{\infty} \cos \frac{5\pi}{6}(2k+1)(2x-1)e^{-5\pi^2\mathcal{D}(k^2+k)/6} = O(e^{-5\pi^2\mathcal{D}/3}).$$

So we introduce the univariate prewavelet

$$\psi_{\mathcal{D}}(x) := \sqrt{\frac{3}{10\pi\mathcal{D}}}\, e^{5\pi^2\mathcal{D}/24} \sum_{m\in\mathbf{Z}} (-1)^m e^{-m^2/5\mathcal{D}} \phi_{\mathcal{D}}(2x-1-m) \tag{4.1}$$

and its perturbation, the approximate prewavelet

$$\tilde{\psi}_{\mathcal{D}}(x) := e^{-(2x-1)^2/6\mathcal{D}} \cos \frac{5\pi}{6}(2x-1), \tag{4.2}$$

differing by

$$|\psi_{\mathcal{D}}(x) - \tilde{\psi}_{\mathcal{D}}(x)| \le (1+\epsilon)e^{-(2x-1)^2/6\mathcal{D}} e^{-5\pi^2\mathcal{D}/3}, \qquad \epsilon \ll 1,$$

such that for any $\varphi = \sum a_m \psi_{\mathcal{D}}(x-m)$ the estimate

$$\|\varphi - \sum a_m \tilde{\psi}_{\mathcal{D}}(x-m)\|_2 \le \frac{\varepsilon_{\mathcal{D}}^2}{2d} \|\varphi\|_2 \tag{4.3}$$

holds. Consequently, in numerical computations with the precision $\varepsilon_{\mathcal{D}}$ one can use both formulas of the prewavelet with the same rights.

In order to define multivariate wavelet bases we formulate the following assertions which can be easily checked by using the Fourier transform of the prewavelet,

$$\mathcal{F}\psi_{\mathcal{D}}(\lambda) = \sqrt{\frac{3\pi\mathcal{D}}{8}}\, e^{5\pi^2\mathcal{D}/24} e^{-\pi i\lambda} e^{-\pi^2\mathcal{D}\lambda^2/4} \sigma_{5\mathcal{D}}\left(\frac{\lambda+1}{2}\right), \tag{4.4}$$

where σ_α denotes the positive and 1-periodic function

$$\sigma_\alpha(\lambda) = \frac{1}{\sqrt{\alpha\pi}} \sum_{m\in\mathbf{Z}} e^{-m^2/\alpha} e^{2\pi i m\lambda} = \sum_{j\in\mathbf{Z}} e^{-\alpha\pi^2(\lambda+j)^2}.$$

LEMMA 4.1. 1. *The half-shifts of the prewavelet* $\{\psi_{\mathcal{D}}(\cdot - m/2)\}_{m\in\mathbf{Z}}$ *are an* L_2*-stable basis in the scaled space* V_1.

2. *The integer shifts of the prewavelet* $\psi_{\mathcal{D}}$ *and of the function*

$$\tilde{\phi}_{\mathcal{D}}(x) := \frac{2}{\sqrt{3\pi\mathcal{D}}} \sum_{m\in\mathbf{Z}} e^{-m^2/3\mathcal{D}} \phi_{\mathcal{D}}(2x-m) \in V_1 \tag{4.5}$$

form an L_2*-stable basis in* V_1.

TABLE 1
Moments of $\psi_{\mathcal{D}}$

Moment	$\mathcal{D} = 2$	$\mathcal{D} = 3$	$\mathcal{D} = 4$	$\mathcal{D} = 5$
0	5.103×10^{-9}	2.143×10^{-13}	8.482×10^{-18}	3.251×10^{-22}
1	-2.551×10^{-9}	-1.071×10^{-13}	-4.241×10^{-18}	-1.626×10^{-22}
2	-3.059×10^{-7}	-2.920×10^{-11}	-2.065×10^{-15}	-1.240×10^{-19}
3	4.601×10^{-7}	4.386×10^{-11}	3.100×10^{-15}	1.862×10^{-19}
4	1.616×10^{-6}	3.685×10^{-9}	4.759×10^{-13}	4.533×10^{-17}
5	-4.116×10^{-6}	-9.287×10^{-9}	-1.195×10^{-12}	-1.134×10^{-16}
6	-7.304×10^{-4}	-4.263×10^{-7}	-1.032×10^{-10}	-1.582×10^{-14}
7	2.701×10^{-3}	1.525×10^{-6}	3.655×10^{-10}	5.577×10^{-14}
8	2.768×10^{-2}	4.460×10^{-5}	2.096×10^{-8}	5.255×10^{-12}

In addition to the fast decay of $\psi_{\mathcal{D}}$ we are interested in the moments of the prewavelet. Since this function is orthogonal to the integer shifts of the Gaussian $\phi_{\mathcal{D}}$, which approximate polynomials very accurately, as seen in Section 2, one can expect that even higher moments are very small and decrease if $\mathcal{D}$ increases. Using the Fourier transform (4.4) we have computed the first eight moments of $\psi_{\mathcal{D}}$, which are contained in Table 1.

5. APPROXIMATE MULTIVARIATE WAVELET DECOMPOSITION

Now we are in a position to discuss the approximate wavelet decomposition of $\mathbf{V}_n$ in the multivariate case. First we introduce an L_2-stable basis in the wavelet space $\mathbf{W}_0$. Consider the function

$$\Psi_{\mathcal{D}}(\mathbf{x}) = \left(\frac{3}{10\pi\mathcal{D}}\right)^{d/2} e^{5\pi^2 d\mathcal{D}/24} \sum_{\mathbf{m}\in\mathbf{Z}^d} (-1)^{|\mathbf{m}|} e^{-|\mathbf{m}|^2/5\mathcal{D}} e^{-|2\mathbf{x}-\mathbf{m}|^2/\mathcal{D}}, \tag{5.1}$$

which represents the tensor product of the univariate functions $\psi_{\mathcal{D}}(x_j - \frac{1}{2})$, $j = 1, \ldots, d$, and its perturbation

$$\tilde{\Psi}_{\mathcal{D}}(\mathbf{x}) = \prod_{j=1}^{d} \tilde{\psi}_{\mathcal{D}}(x_j - \tfrac{1}{2}) = \phi_{3\mathcal{D}}(\mathbf{x}) \prod_{j=1}^{d} \cos\frac{5\pi}{3} x_j. \tag{5.2}$$

From (4.3) it is clear that within the precision $\varepsilon_{\mathcal{D}}$ these two functions can be identified. By Lemma 4.1, the scaled space $\mathbf{V}_1$ is spanned by the half-shifts $\{\Psi_{\mathcal{D}}(\cdot - \mathbf{m}/2)\}_{\mathbf{m}\in\mathbf{Z}^d}$ and consequently the set

$$\{\Psi_{\mathcal{D}}(\cdot - \mathbf{v} - \mathbf{m}) : \mathbf{m} \in \mathbf{Z}^d,\ \mathbf{v} \in \mathcal{V}'\}$$

is an L_2-stable basis in $\mathbf{W}_0$. Here again $\mathscr{V}$ denotes the set of vertices of the cube $[0, \frac{1}{2}]^d$ and $\mathscr{V}' = \mathscr{V} \setminus \{\mathbf{0}\}$. Thus we obtain the L_2-stable basis

$$\{\phi_{\mathscr{D}}(\cdot - \mathbf{m}) : \mathbf{m} \in \mathbf{Z}^d\}, \qquad \{\Psi_{\mathscr{D}}(2^j \cdot - \mathbf{v} - \mathbf{m}) : \mathbf{m} \in \mathbf{Z}^d, \mathbf{v} \in \mathscr{V}', j = 0, \ldots, n-1\}$$

in $\tilde{\mathbf{V}}_n$, which is by Theorem 3.1 almost the space $\mathbf{V}_n$.

The elements of this basis have the property that the application of important integral operators can be given analytically. In addition to series representations obtained from (5.1), efficient expressions can be derived by using the perturbed basis function (5.2). For example,

$$\mathscr{N}\tilde{\Psi}_{\mathscr{D}}(\mathbf{x}) = \frac{1}{4\pi}\int_{\mathbf{R}^3} \frac{e^{-|\mathbf{y}|^2/3\mathscr{D}}}{|\mathbf{x}-\mathbf{y}|} \prod_{j=1}^{3} \cos\frac{5\pi}{3} y_j \, d\mathbf{y} = \frac{1}{8}\sum_{|\mu_j|=1} \frac{1}{4\pi}\int_{\mathbf{R}^3} \frac{e^{-|\mathbf{y}|^2/3\mathscr{D}}}{|\mathbf{x}-\mathbf{y}|} e_{\mathbf{y}}\left(\frac{5\boldsymbol{\mu}}{6}\right) d\mathbf{y}$$

with $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3) \in \mathbf{R}^3$. Using the Fourier transforms one obtains

$$\begin{aligned}\frac{1}{4\pi}\int_{\mathbf{R}^3} \frac{e^{-|\mathbf{y}|^2/3\mathscr{D}}}{|\mathbf{x}-\mathbf{y}|} e_{\mathbf{y}}\left(\frac{5\boldsymbol{\mu}}{6}\right) d\mathbf{y} &= (3\pi\mathscr{D})^{3/2}\int_{\mathbf{R}^3} \frac{e^{-3\pi^2\mathscr{D}|\boldsymbol{\lambda}-5\boldsymbol{\mu}/6|^2}}{4\pi^2|\boldsymbol{\lambda}|^2} e_{\boldsymbol{\lambda}}(\mathbf{x})\, d\boldsymbol{\lambda} \\ &= 3\mathscr{D}\pi^{3/2} e^{-25\pi^2\mathscr{D}/4}\int_{\mathbf{R}^3} \frac{e^{-\pi^2|\boldsymbol{\lambda}|^2}}{4\pi^2|\boldsymbol{\lambda}|^2} e_{\boldsymbol{\lambda}/\sqrt{3\mathscr{D}}}\left(\mathbf{x} - \frac{5\pi\mathscr{D} i}{2}\boldsymbol{\mu}\right) d\boldsymbol{\lambda}.\end{aligned}$$

Since the integral

$$\pi^{3/2}\int_{\mathbf{R}^3} \frac{e^{-\pi^2|\boldsymbol{\lambda}|^2}}{4\pi^2|\boldsymbol{\lambda}|^2} e_{\boldsymbol{\lambda}}(\mathbf{x})\, d\boldsymbol{\lambda} = \frac{1}{2|\mathbf{x}|}\int_0^{|\mathbf{x}|} e^{-t^2} dt, \qquad \mathbf{x} \in \mathbf{R}^3$$

(see [16]), is an analytic function we derive

$$\frac{1}{4\pi}\int_{\mathbf{R}^3} \frac{e^{-|\mathbf{y}|^2/3\mathscr{D}}}{|\mathbf{x}-\mathbf{y}|} e_{\mathbf{y}}\left(\frac{5\boldsymbol{\mu}}{6}\right) d\mathbf{y} = \frac{3\mathscr{D} e^{-25\pi^2\mathscr{D}/4}}{2}\int_0^1 e^{-zt^2} dt,$$

where

$$z = \frac{1}{3\mathscr{D}}\sum_{j=1}^{3}\left(x_j - \frac{5\pi\mathscr{D} i}{2}\mu_j\right)^2 = \frac{|\mathbf{x}|^2}{3\mathscr{D}} - \frac{25\pi^2\mathscr{D}}{4} - \frac{5\pi i}{3}\langle \mathbf{x}, \boldsymbol{\mu}\rangle.$$

Thus we get the one-dimensional integral

$$\mathscr{N}\tilde{\Psi}_{\mathscr{D}}(\mathbf{x}) = \frac{3\mathscr{D}}{2}\int_0^1 \exp\left(-\frac{|\mathbf{x}|^2 t^2}{3\mathscr{D}} + \frac{25\pi^2\mathscr{D}(t^2-1)}{4}\right)\prod_{j=1}^{3}\cos\frac{5\pi}{3} y_j t^2 dt,$$

which can be expressed also by the error function of complex argument.

Now we consider the problem of finding the approximate wavelet decomposition of a given element belonging to $\mathbf{V}_n$ and to prove estimate (1.16). Following Theorem 3.1 one has to determine the orthogonal projections onto $\mathbf{V}_0$ and $\mathbf{W}_j$.

Since $\phi_{\mathcal{D}}(\mathbf{x}) = \Pi\, \phi_{\mathcal{D}}(x_j)$ the orthogonal projections P_0 onto $\mathbf{V}_0$ can be given as the tensor product

$$P_0 = R_0 \otimes \cdots \otimes R_0 \tag{5.3}$$

of the univariate L_2-projections R_0 onto V_0. This mapping is described in the following lemma, which will be proved in the final section.

LEMMA 5.1. *The orthogonal projection R_0 onto V_0 has the form*

$$R_0 f = \sum_{k\in\mathbf{Z}} (f, \check{\phi}_{\mathcal{D}}(\cdot - k))_2 \phi_{\mathcal{D}}(\cdot - k), \tag{5.4}$$

where the function $\check{\phi}_{\mathcal{D}} \in V_0$ is given by the formula

$$\check{\phi}_{\mathcal{D}}(x) = \sum_{k\in\mathbf{Z}} a_k(\mathcal{D}) \phi_{\mathcal{D}}(x-k)$$

with the coefficients

$$a_k(\mathcal{D}) = (-1)^k c(\mathcal{D}) e^{k^2/2\mathcal{D}} \sum_{r=|k|}^{\infty} (-1)^r e^{-(r+1/2)^2/2\mathcal{D}}$$

and the constant

$$c(\mathcal{D}) = \frac{1}{2\pi^2\mathcal{D}^2} \left(\sum_{j\in\mathbf{Z}} (-1)^j j e^{-\pi^2\mathcal{D}(2j+1)^2/2} \right)^{-1}.$$

Let us turn to the orthogonal projections Q_j onto $\mathbf{W}_j$. Since these spaces are derived by scaling from $\mathbf{W}_0$ it suffices to consider the operator Q_0. We will construct a mapping $\tilde{Q}_0$ onto $\mathbf{W}_0$ such that $\|\tilde{Q}_0 - Q_0\|_2 \le c\varepsilon_{\mathcal{D}}$ with some constant c depending only on the dimension d. To this end we consider the basis in $\mathbf{W}_0$ obtained by the simple tensor product approach for the construction of multivariate prewavelets (see [17]). Denoting

$$w_0(x) := \tilde{\phi}_{\mathcal{D}}(x), \qquad w_{1/2}(x) := \psi_{\mathcal{D}}(x) \tag{5.5}$$

(see Lemma 4.1), we introduce the collection of functions

$$\Phi_{\mathbf{v}}(\mathbf{x}) = w_{v_1}(x_1) \cdots w_{v_d}(x_d) \in \mathbf{W}_0, \qquad \mathbf{v} \in \mathcal{V}'.$$

Since the function w_0 is the right-hand side of the approximate refinement equation (2.7) for $\phi_{\mathcal{D}}$ it follows from Lemma 4.1 that the set $\{\Phi_{\mathbf{v}}(\cdot - \mathbf{m})\}_{\mathbf{m}\in\mathbf{Z}^d,\mathbf{v}\in\mathcal{V}'}$ is an L_2-stable basis in $\mathbf{W}_0$. Moreover, if in the definition of $\Phi_{\mathbf{v}}$ the function w_0 is replaced by the Gaussian function $\phi_{\mathcal{D}}(x)$, then the corresponding principal shift invariant spaces

$$X_{\mathbf{v}} := \{\Phi_{\mathbf{v}}(\cdot - \mathbf{m}), \mathbf{m} \in \mathbf{Z}^d\}$$

are orthogonal, $X_{\mathbf{v}} \perp X_{\mathbf{v}'}$ for $\mathbf{v} \neq \mathbf{v}'$. Thus the orthogonal projection onto $\bigoplus_{\mathbf{v}\in\mathcal{V}'} X_{\mathbf{v}}$ is

$$\bar{Q}_0 := \sum_{\mathbf{v}\in\mathcal{V}'} R_{v_1} \otimes \cdots \otimes R_{v_d} \tag{5.6}$$

with the univariate projections $R_0: L_2(\mathbf{R}) \to V_0$ and $R_{1/2}: L_2(\mathbf{R}) \to W_0$. It is evident that

$$\|\bar{Q}_0 - Q_0\| \leq c\varepsilon_{\mathcal{D}}, \tag{5.7}$$

where c depends on d only. To derive a mapping into $\mathbf{W}_0$ we introduce the small perturbation of the operator $\bar{Q}_0$,

$$\tilde{Q}_0 := \sum_{\mathbf{v}\in\mathcal{V}'} \tilde{R}_{v_1} \otimes \cdots \otimes \tilde{R}_{v_d}, \tag{5.8}$$

where we define (see (5.4) and (5.5))

$$\tilde{R}_0 f = \sum_{k\in\mathbf{Z}} (f, \check{\phi}_{\mathcal{D}}(\cdot - k))_2 w_0(\cdot - k),$$

and the mapping $\tilde{R}_{1/2}$ is described in the following lemma, which will be proved in the next section.

LEMMA 5.2. *The orthogonal projection $R_{1/2}$ onto W_0 can be approximately represented in the form*

$$\tilde{R}_{1/2} f = \sum_{k\in\mathbf{Z}} (f, \check{\psi}_{\mathcal{D}}(\cdot - k))_2 \psi_{\mathcal{D}}(\cdot - k),$$

where the function $\check{\psi}_{\mathcal{D}}$ is given as the sum

$$\check{\psi}_{\mathcal{D}}(x) := \sum_{m\in\mathbf{Z}} a_k \psi_{\mathcal{D}}(x - k) \in W_0,$$

with the coefficients

$$a_k = e^{k^2/3\mathcal{D}}\Big(c_0 \sum_{j=|k|}^{\infty} (-1)^j e^{-3(j+1/2)^2/\mathcal{D}} + (-1)^k c_1 \sum_{j=|k|}^{\infty} \big(e^{-3(j+1/4)^2/\mathcal{D}} - e^{-3(j+3/4)^2/\mathcal{D}}\big)\Big), \tag{5.9}$$

and the numbers c_0 *and* c_1 *equal to*

$$c_0 = \frac{4}{3\pi^2\mathcal{D}^2}\left(\sum_{j\in\mathbf{Z}} (-1)^j (6j+1) e^{-\pi^2\mathcal{D}(6j+1)^2/12}\right)^{-1},$$

$$c_1 = \frac{2}{3\pi^2\mathcal{D}^2}\left(\sum_{j\in\mathbf{Z}} (-1)^j (6j+1) e^{-\pi^2\mathcal{D}(6j+1)^2/3}\right)^{-1}.$$

There exists a constant c such that

$$\|R_{1/2} f - \tilde{R}_{1/2} f\|_2 \le c e^{-\pi^2\mathcal{D}} \|f\|_2, \qquad \forall f \in L_2(\mathbf{R}). \tag{5.10}$$

From Lemma 5.2, the refinement equation (2.7), and (5.7) it is clear that we have

$$\|\tilde{Q}_0 - Q_0\| \le c\varepsilon_{\mathcal{D}}, \tag{5.11}$$

with some constant c depending only on d. Thus we obtain the following approximate wavelet decomposition of the space $\mathbf{V}_n$.

THEOREM 5.1. *There exists a constant c depending on the space dimension d and on n such that for any* $\varphi_n \in \mathbf{V}_n$ *the estimate*

$$\|\varphi_n - \sum_{j=-1}^{n-1} \tilde{Q}_j \varphi_n\|_2 \le c\varepsilon_{\mathcal{D}} \|\varphi_n\|_2$$

holds, where $\tilde{Q}_{-1} = P_0$ *is defined in* (5.3) *and the mappings* $\tilde{Q}_j$ *onto* $\mathbf{W}_j$ *are obtained by scaling from* $\tilde{Q}_0$ *given in* (5.8).

6. THE ORTHOGONAL PROJECTIONS R_0 AND $R_{1/2}$

The construction of the orthogonal projections onto V_0 and W_0 uses some well-known facts about principal shift invariant spaces (see [4]), which we recall briefly. Denote by $S(\eta)$ the L_2-closure of finite linear combinations of the shifts $\eta(\cdot - m)$, $m \in \mathbf{Z}$, of a generating function η and suppose that the shifts form an L_2-stable basis of $S(\eta)$. Note that for any

$$\phi(x) = \sum_{m\in\mathbf{Z}} a_m \eta(x-m) \in S(\eta)$$

we have

$$\mathscr{F}\phi(\lambda) = \tau(\lambda)\mathscr{F}\eta(\boldsymbol{\lambda}) \qquad \text{with } \tau(\lambda) := \sum_{m\in\mathbf{Z}} a_m e_{-m}(\lambda) \in L_2(0, 1).$$

Furthermore if $\mathscr{F}\phi = \tau\mathscr{F}\eta \in L_2(\mathbf{R})$ with a 1-periodic function τ, then

$$\|\phi\|_{L_2(\mathbf{R})} = \|\tau[\mathscr{F}\eta, \mathscr{F}\eta]^{1/2}\|_{L_2(0,1)},$$

where the bracket product stands for the 1-periodic function

$$[f, g] := \sum_{k\in\mathbf{Z}} f(\cdot - k)\overline{g(\cdot - k)}.$$

For each $f \in L_2(\mathbf{R})$ the L_2-projection $Pf \in S(\eta)$ is given by

$$\mathscr{F}(Pf) = \frac{[\mathscr{F}f, \mathscr{F}\eta]}{[\mathscr{F}\eta, \mathscr{F}\eta]}\mathscr{F}\eta.$$

Hence the shifts of the function $\check{\eta} \in S(\eta)$ defined via

$$\mathscr{F}\check{\eta} = \frac{\mathscr{F}\eta}{[\mathscr{F}\eta, \mathscr{F}\eta]} \tag{6.1}$$

form the corresponding biorthogonal basis, i.e., $(\eta(\cdot - k), \check{\eta})_2 = \delta_{0k}$, $k \in \mathbf{Z}$.

Applied to R_0 we have to determine the Fourier coefficients of the inverse function to

$$[\mathscr{F}\phi_{\mathscr{D}}, \mathscr{F}\phi_{\mathscr{D}}] = \pi\mathscr{D}\sum_{m\in\mathbf{Z}} e^{-\pi^2\mathscr{D}(\lambda-m)^2} = \sqrt{\frac{\pi\mathscr{D}}{2}}\sum_{k\in\mathbf{Z}} e^{-k^2/2\mathscr{D}}e^{2\pi ikx},$$

which is closely connected with the theta function.

Lemma 6.1. *Let $a > 0$. Then*

$$\Big(\sum_{k\in\mathbf{Z}} e^{-ak^2}e^{2\pi ikx}\Big)^{-1} = \sum_{k\in\mathbf{Z}} a_k e^{2\pi ikx},$$

where

$$a_k = (-1)^k c e^{ak^2}\sum_{r=|k|}^{\infty}(-1)^r e^{-a(r+1/2)^2}$$

and

$$c^{-1} = \sum_{r=0}^{\infty}(-1)^r(2r+1)e^{-a(r+1/2)^2}.$$

Setting $a^{-1} = 2\mathcal{D}$ we obtain immediately the assertion of Lemma 5.1.

Note that the cardinal interpolant $\eta^* \in S(\eta)$ which satisfies $\eta^*(k) = \delta_{0k}$, $k \in \mathbf{Z}$, has the Fourier transform

$$\mathcal{F}\eta^* := \frac{\mathcal{F}\eta}{[\mathcal{F}\eta, 1]}.$$

Hence Lemma 6.1 gives an explicit formula for the interpolant $Q_h f$ composed of the scaled shifts $\phi_{\mathcal{D}}(\mathbf{x}/h - \mathbf{k})$ with $Q_h f(h\mathbf{m}) = f(h\mathbf{m})$ for all $\mathbf{m} \in \mathbf{Z}^d$,

$$Q_h f(\mathbf{x}) = \sum_{\mathbf{m}\in\mathbf{Z}^d} f(h\mathbf{m}) q_{\mathcal{D}}(\mathbf{x}/h - \mathbf{m}),$$

where

$$q_{\mathcal{D}}(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbf{Z}^d} A_{\mathbf{k}} \phi_{\mathcal{D}}(\mathbf{x} - \mathbf{k}) \qquad \text{with } A_{\mathbf{k}} = a_{k_1} \cdots a_{k_d}$$

and

$$a_k = (-1)^k c \sum_{r=0}^{\infty} (-1)^r e^{-(r+1/2)^2/\mathcal{D}} e^{-|k|(2r+1)/\mathcal{D}},$$

$$c^{-1} = \sum_{r=0}^{\infty} (-1)^r (2r+1) e^{-(r+1/2)^2/\mathcal{D}}.$$

It was proved in [15] that this interpolant approximates with exponential rate up to a saturation error

$$|f(\mathbf{x}) - Q_h f(\mathbf{x})| \leq ((2+4d)e^{-a/2h} + 4de^{-\pi^2\mathcal{D}}) \int_{\mathbf{R}^d} |\mathcal{F}f(\boldsymbol{\lambda})| \exp(a|\boldsymbol{\lambda}|)\, d\boldsymbol{\lambda},$$

if the last integral is finite.

Now we come to the proof of Lemma 5.2. We will construct an almost biorthogonal basis to $\{\psi(\cdot - m)\}_{m\in\mathbf{Z}}$ in the space $W_0 = S(\psi_{\mathcal{D}})$. Following (6.1) one has to find the inverse function of

$$\sum_{m\in\mathbf{Z}} |\mathcal{F}\psi_{\mathcal{D}}(\lambda + m)|^2. \tag{6.2}$$

Using several times the transformation formula (2.4) it can be shown that this function is approximated with a relative error less than $3e^{-5\pi^2\mathcal{D}/2}$ by the function

$$g(\lambda) := \frac{3\pi\mathcal{D}}{4} \sum_{k\in\mathbf{Z}} \left(e^{-3\pi^2\mathcal{D}(k+\lambda+1/6)^2} + e^{-3\pi^2\mathcal{D}(k-\lambda+1/6)^2}\right).$$

Hence it suffices to determine the Fourier coefficients of the 1-periodic function g^{-1}. Note that $g(z)$ is analytic and quasi-doubly periodic with

$$g(z+1) = g(z), \qquad g\left(z + \frac{i}{\pi\mathcal{D}}\right) = -e^{3/\mathcal{D}}e^{6\pi iz}g(z); \tag{6.3}$$

therefore in any rectangle $[z, z+1) \times [z, z+i/\pi\mathcal{D})$ there are three zeros of $g(z)$. It can easily be seen that in $[-\frac{1}{4}, \frac{3}{4}) \times [-\frac{1}{4}, -\frac{1}{4} + i/\pi\mathcal{D})$ the zeros are at $z_0 = (0, i/2\pi\mathcal{D})$ and near the points $(\frac{1}{2}, i/4\pi\mathcal{D})$ and $(\frac{1}{2}, 3i/4\pi\mathcal{D})$. More precisely, we have

$$g(\tfrac{1}{2} + iy) = \frac{3\pi\mathcal{D}}{2} e^{3\pi^2\mathcal{D}y^2} \sum_{k=0}^{\infty} \Big(\cos 2\pi^2\mathcal{D}y(3k+1)e^{-\pi^2\mathcal{D}(3k+1)^2/3} + \cos 2\pi^2\mathcal{D}y(3k+2)e^{-\pi^2\mathcal{D}(3k+2)^2/3}\Big),$$

such that $g(\frac{1}{2} + \; + iy) = 0$ at the points

$$z_1 = \frac{1}{2} + \frac{i}{4\pi\mathcal{D}} - i\epsilon \qquad \text{and} \qquad z_2 = \frac{1}{2} + \frac{3i}{4\pi\mathcal{D}} + i\epsilon \text{ with } 0 < \epsilon < e^{-\pi^2\mathcal{D}}. \tag{6.4}$$

To determine the Fourier coefficients we apply (6.3) and the Residue Theorem to the integral

$$a_k = \int_{-1/4}^{3/4} \frac{e^{-2\pi ikx}}{g(x)}\,dx$$

and obtain the recurrence relation

$$a_k = -e^{(2k-3)/\mathcal{D}}a_{k-3} + (r_0 + r_1 + r_2), \tag{6.5}$$

where

$$r_j = 2\pi i \operatorname*{Res}_{z=z_j} \frac{e^{-2\pi ikz}}{g(z)}.$$

It is easy to see that

$$r_0 = c_0 e^{k/\mathcal{D}} e^{-3/4\mathcal{D}}, \qquad c_0 = \frac{4}{3\pi^2\mathcal{D}^2}\left(\sum_{j\in\mathbf{Z}} (-1)^j(6j+1)e^{-\pi^2\mathcal{D}(6j+1)^2/12}\right)^{-1}. \tag{6.6}$$

Further, the residues r_1 and r_2 are

$$r_1 = -(-1)^k c_1 e^{k/2\mathcal{D}} e^{-3/16\mathcal{D}} e^{-\pi\epsilon(2k-3/2)} e^{-3\pi^2\mathcal{D}\epsilon^2},$$
$$r_2 = (-1)^k c_1 e^{3k/2\mathcal{D}} e^{-27/16\mathcal{D}} e^{\pi\epsilon(2k-3/2)} e^{-3\pi^2\mathcal{D}\epsilon^2},$$

with the constant

$$c_1 = \frac{2}{3\pi^2\mathcal{D}^2}\left(\sum_{j\in\mathbf{Z}} (3j+1)\sin\frac{\pi}{2}(1-4\pi\mathcal{D}\epsilon)(3j+1)e^{-\pi^2\mathcal{D}(3j+1)^2/3}\right)^{-1}. \qquad (6.7)$$

It turns out, that (6.5) has a unique solution $\{a_k\} \in l_2(\mathbf{Z})$ which is given by

$$a_k = e^{k^2/3\mathcal{D}} \Big(c_0 \sum_{j=|k|}^{\infty} (-1)^j e^{-3(j+1/2)^2/\mathcal{D}}$$
$$+ (-1)^k c_1 \sum_{j=|k|}^{\infty} \big(e^{3\pi\epsilon/2} e^{-3(j+1/4-\pi\mathcal{D}\epsilon)^2/\mathcal{D}} - e^{-3\pi\epsilon/2} e^{-3(j+3/4+\pi\mathcal{D}\epsilon)^2/\mathcal{D}}\big)\Big). \qquad (6.8)$$

So we have derived the Fourier coefficients of g^{-1}, i.e., up to a relative error less than $3e^{-5\pi^2\mathcal{D}/2}$ the coefficients of the biorthogonal basis function to the wavelet basis $\psi_{\mathcal{D}}(\cdot - k)$. Furthermore, since $0 < \epsilon < e^{-4\pi^2\mathcal{D}/3}$ (see (6.4)) we may set in (6.7) and (6.8) the small number $\epsilon = 0$ and obtain together with (6.6) formula (5.9) of Lemma 5.2. In doing so we make an error for the Fourier coefficients, which is less than $e^{-\pi^2\mathcal{D}}|a_k|$. In view of the rapid decay of the wavelet the estimate (5.10) follows immediately, which completes the proof of Lemma 5.2.

7. CONCLUDING REMARKS

The decomposition indicated in Theorem 5.1 was implemented in the one-dimensional case to obtain compressed representations for density functions and applied to the computation of one-dimensional integral operators. Table 2 provides some numerical results concerning evaluation of the Hilbert transform of different functions given on $(-500, 500)$. Originally the functions are approximated by the quasi-interpolant

$$u_h(x) := \mathcal{D}^{-n/2} \sum_{m=-32{,}000}^{32{,}000} u(hm)\exp\left(-\frac{(x-hm)^2}{\mathcal{D}h^2}\right) \in V_6 \qquad (7.1)$$

with $h = 1/64$, i.e., determined by 64,001 point values. Table 2 lists the number of all basis functions in V_0 and W_j, $j = 0, \ldots, 5$, necessary to compute the Hilbert transform at all grid points $\{mh\}$ with the prescribed accuracy $\varepsilon_{\mathcal{D}}$.

Except $\exp(-x^2/1000)$ the considered density functions φ are either nonsmooth or oscillating, such that the support of the projections onto the spaces V_0 and W_j, $Q_j\varphi$, is

TABLE 2
Number of Basis Functions Necessary for Computing the Hilbert Transform at All Grid Points with Prescribed Accuracy $\varepsilon_{\mathcal{D}}$

Function	$\mathcal{D} = 2$	$\mathcal{D} = 3$	$\mathcal{D} = 4$
1	2563	3509	4689
$\lvert 500 - x \rvert$	2290	3174	4286
$\sin(\pi x)$	7624	7897	8116
$\exp(-x^2/1000)$	811	897	1085
$\exp(-x^2/1000)\sin(\pi x)$	3826	4140	4354

larger than the interval $(-500, 500)$. Nevertheless the table shows that it is possible to obtain significant compression rates for data if the density function behaves sufficiently well. The same applies to the computing time. For example, to evaluate one point value of the integral besides the Hilbert transform of all Gaussians $\exp(-(x-m)^2/\mathcal{D})$, which form $Q_{-1}\varphi \in V_0$, one has to compute only the Hilbert transform of the wavelets with essential support near this point. Therefore the number of summands necessary to compute the Hilbert transform at one given point is essentially smaller than 64,001, necessary if the representation (7.1) is used, and even much smaller than the number given in Table 2.

We believe that the compression rates can be improved further if one tries to approximate the finest scale space V_n instead by a nearly orthogonal sum of wavelet spaces by another direct sum of spaces spanned by approximate biorthogonal wavelets. It follows from Lemma 5.1 that for compactly supported φ the coefficients c_m of the orthoprojection

$$Q_{-1}\varphi = \sum_{m=-\infty}^{\infty} c_m \exp\left(-\frac{(x-m)^2}{\mathcal{D}}\right) \in V_0$$

decay asymptotically like $|c_m| \asymp \exp(-\rho_m/2\mathcal{D})$, with $\rho_m = \inf|m-x|$, $x \in \operatorname{supp}\varphi$ and a rather large constant. By using other projection operators onto V_0 it is possible to derive a faster decay of the corresponding coefficients, such that one has to store fewer data. This construction of biorthogonal wavelets by using the idea of approximate multiresolution analysis as well as the construction of multivariate approximate wavelets different from tensor products is a topic of future research.

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